BESSEL'S EQUATION AND ITS SOLUTION: BESSEL'S POLYNOMIAL

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Solution Of Bessel's Equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \quad \longrightarrow \quad (1)$$

- Let $y=\sum_{r=0}^{\infty} a_r x^{m+r}$ is a solution of the above equ.
- **So that,** $$\frac{dy}{dx}=\sum_{r=0}^{\infty} a_r(m+r)x^{m+r-1}$$
- **and** $$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r(m+r)(m+r-1)x^{m+r-2}$$

Solution Of Bessel's Equation

- Substituting this in equ(1),

$$x^2\sum_{r=0}^{\infty} a_r(m+r)(m+r-1)x^{m+r-2} + x\sum_{r=0}^{\infty} a_r(m+r)x^{m+r-1} + (x^2-n^2)\sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

- Rearranging this we will get,

$$\sum_{r=0}^{\infty} a_r[(m+r)^2 - n^2]x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0 \quad (2)$$

Indicial Equation Of Bessel's Equation

- Equating the co-efficient of lowest power term of x in equ(2), with zero, and putting r = 0,

$$a_0[(m+0)^2 - n^2] = 0$$

- Therefore, $m^2 = n^2$ i.e $m = \pm n$
- Equating co-efficient of next lowest power term (x^{m+1}) with zero and putting r=1,

$$a_1[(m+1)^2 - n^2] = 0$$

Since $[(m+1)^2 - n^2] \neq 0$ $\therefore a_1 = 0$

Recursion Relation

- Equating the co-efficient of x^{m+r+2} in equ(2) with zero,

$$a_{r+2}\left[(m+r+2)^2-n^2\right]+a_r=0$$

$$a_{r+2}=-\frac{1}{\left[(m+r+2)^2-n^2\right]}a_r$$

- Since $a_1=0$,Therefore, $a_3=a_5=a_7=0$
- Now, $for\ r=0,\quad a_2=\frac{1}{\left[(m+2)^2-n^2\right]}a_0$

$$for\ r=2, a_4=\frac{1}{\left[(m+4)^2-n^2\right]}a_2$$

$$=\frac{1}{\left[(m+4)^2-n^2\right]\left[(m+2)^2-n^2\right]}a_0 \quad and\ so\ on.$$

General Solution

- Substituting the values of all co-efficients ,

$$y = a_0 x^m [1 - \frac{1}{(m+2)^2 - n^2} x^2 + \frac{1}{[(m+4)^2 - n^2][(m+2)^2 - n^2]} x^4 - \cdots]$$

- For m = n,

$$y = a_0 x^n [1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2 . 2! \, (n+1)(n+2)} x^4 - \cdots]$$

$$y = a_0 x^n \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2^{2r} . r! \, (n+1)(n+2) \ldots \ldots (n+r)}$$

Bessel Polynomial

- Now if we put , $a_0 = \frac{1}{2^n \Gamma(n+1)}$

$$y = \frac{1}{2^n \Gamma(n+1)} \sum_{r=0}^{\infty} (-1)^r \frac{x^{n+2r}}{2^{2r}.\, r!\,(n+1)(n+2)\ldots\ldots(n+r)} = J_n(x)$$

$$J_n(x) = \left(\frac{x}{2}\right)^n \{\frac{1}{\Gamma(n+1)} - \frac{1}{1!\,\Gamma(n+2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma(n+4)}\left(\frac{x}{2}\right)^6 + \cdots\}$$

- Using $\Gamma(n+1) = n!$ we get,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r}$$

Bessel Polynomial

- Now if we put , n=0, we get,

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{x}{2}\right)^{2r} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots$$

- If we put n=1, $J_1(x) = \frac{x}{2} - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} + \cdots$

Zeroes of Bessel Function : Bourget's Hypothesis

- When the function $J_n(x)$ are plotted on the same graph paper, though, none of the zeroes seem to coincide for different values of n except for the zero at x=0.